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INSTABILITY OF PLANAR OSCILLATIONS IN A CERTAIN NON-LINEAR SYSTEM UNDER RANDOM EXCITATION

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Parametric resonance has been proposed in reference [1] as a model for transition from planar to whirling vibrations of bars and beams with loose supports (such as underwater piles, heat exchanger tubes with gaps in support plates etc.). A single-mass t.d.o.f. system is considered, with motion of mass being described by its Cartesian co-ordinates $X(t)$ and $Y(t)$. The system is at equilibrium in the origin $X = 0$, $Y = 0$, and its springs provide a radially oriented restoring force F, its components in the X and Y directions being $F(X)$ and $F(Y/r)$, respectively, where $r = \sqrt{X^2 + Y^2}$. The equations of motion are written as

$$m\ddot{X} + F(r)(1 + \mu)(X/r) = 0, \qquad m\ddot{Y} + F(r)(Y/r) = m\varsigma(t), \tag{1, 2}$$

where μ is a (small) parameter of asymmetry of the restoring force, or detuning parameter between natural frequencies of small oscillations in the X and Y directions. In case of a linear restoring force, where $F(r) = Kr$, equations (1) and (2) are seen to be uncoupled, so that solution to equation (1), in case of zero initial conditions (ICs), is just $X \equiv 0$. This is the case of a planar motion $Y(t)$, excited by the external force $\varsigma(t)$ and/or by non-zero initial conditions for $Y(t)$.

If, however, the restoring force $F(r)$ contains a hardening non-linearity, coupling between equations (1) and (2) appears. The one-dimensional motion $X = 0$, $Y = r$ may then become unstable in X, as long as the directly excited periodic motion $Y(t)$, as governed by equation (2) with $r = Y$, appears in equation (1) as a parametric excitation. The phenomenon resembles somewhat the classical autoparametric resonance [2], but with an important difference: the parametric instability is provided in the present case by higher harmonics only of $Y(t)$, and as shown above, the effect does not exist for a linear $F(r)$.

Instability of the planar motion has been studied in reference [1] for the case of periodic-in-time $Y(t)$, both for free (undamped) case with non-zero ICs for Y and for the case of forced oscillations, with $\varsigma(t) = \Lambda \sin \omega t$ in equation (2). Equation (1), linearized in the vicinity of the equilibrium state $X = 0$, is reduced then to the Mathieu equation for each of the harmonics of $Y(t)$. The analytical solution has been obtained first by Krylov–Bogoliubov (KB) averaging [3], for the conservative case with a small cubic non-linearity in the $F(r)$. It has shown the system to be exactly at the stability boundary in the important special case of a perfect axial symmetry ($\mu = 0$). It was decided then to obtain a benchmark exact solution for such a "doubtful" case by using the following specific form for the restoring force,

$$F(r) = (m\Omega^2/k)(\tan kr/\cos^2 kr), \tag{3}$$

where k is an arbitrary parameter of non-linearity, and Ω is clearly seen to be the natural frequency of small (linear) oscillations in the Y direction: if $X = 0$, $|Y| = r$, then

$F(r) \to m\Omega^2 r$ when $kr \to 0$. The next, or two-term power series approximation for $F(r)$ in kr yields a cubic non-linearity. As for the other extreme of large-amplitude displacements, it can be seen that the motion is confined within an ellipse with axes $R = \pi/2$ and $R = (1 + \mu)(\pi/2)$, $R = 1/k$. Therefore, the model should be adequate to describe rattling motions within loose support of circular or slightly non-circular shape.

Equation (2) with zero RHS and expression (3) for $F(r)$, with $r = Y$, has an exact solution for $Y(t)$, as discovered originally in reference [4]. Expanding this solution into Fourier series and combining it with the available data on stability boundaries for the Mathieu equation [5] resulted in boundaries of the instability domain in the plane A, μ, where A is the response amplitude in $Y(t)$, as governed by the initial conditions [1]. One of the branches of the boundary was found to be a tangent to the axis $\mu = 0$ at the origin. This result clearly correlates with the approximate one as obtained (for small A's) by the KB averaging. And the most important conclusion of the exact solution is the fact, that with increasing A the instability domain moves away eventually from the ordinate axis, i.e., *the perfectly symmetric system belongs to the interior of the stability domain* rather than to its boundary. The possible implication of this result is the necessary to exceed a certain threshold level of the external excitation if the latter is the source of vibrations in the Y direction rather than the initial displacement and/or velocity.

This expectation had been confirmed in reference [1] by an approximate analytical solution, by KB-averaging, for the case of the sinusoidal-in-time excitation in the Y direction (with the non-zero RHS in equation (2)) for a perfectly symmetric system ($\mu = 0$). The same conclusion has been attained through a direct numerical integration of equation (2) and linearized equation (1). The "transmitted Ince–Strutt chart" has been calculated, i.e., stability chart for equilibrium in the X direction, in terms of the amplitude and frequency of excitation in the Y direction.

In this note, the case of zero-mean white-noise random excitation $\varsigma(t)$ with intensity D in the Y direction is considered, this kind of excitation being typical for the above-mentioned potential applications. The basic goal is to find the threshold intensity of excitation, which corresponds to excitation of vibration in the (normal) X direction. The results may be of importance for design, since dynamic instability of the equilibrium state $X = 0$ implies two-dimensional motion of the whirling type indeed. This can be seen from numerical integration data [1] for the full system (1) and (2) with zero RHS for a certain (asymmetric) case of instability; violent tangential motions are involved in this whirling, which may lead to a greatly increased wear in the loose support. The analytical study is very difficult, as long as in the case of a lightly damped system, with similar damping ratios in two directions, the parametric random excitation $Y(t)$, as applied to system (1), is narrow-band even in case of a broadband external excitation, with bandwidth being of the same order as that of system (1). Thus, a direct Monte-Carlo simulation is used, for the case of a perfect axial symmetry ($\mu = 0$). Viscous damping terms are also added to the basic equations, so that a stationary response $Y(t)$ may exist.

Introducing non-dimensional variables $x = kX = X/R$, $y = kY = Y/R$, equations (1) and (2) (with $r \cong Y$ as the linearization condition and with viscous damping terms added) may be reduced, respectively, to

$$\ddot{x} + 2\beta\dot{x} + g(y(\tau))x = 0, \qquad g(y) = f(y)/y, \quad f(y) = \Omega^2(\tan y/\cos^2 y), \tag{4}$$

$$\ddot{y} + 2\alpha\dot{y} + f(y) = k\varsigma(t). \tag{5}$$

The procedure for establishing a stability threshold for system (1) and (2) was essentially as follows. Numerical integration of the equations with computer-generated white noise $\varsigma(t)$ was performed with zero ICs within time interval T with the duration being about 5000

TABLE 1

Threshold excitation intensity for instability of planar response

α/Ω	0·01	0·03	0·05	0·1
σ_*	0·4	0·75	0·962	1·4

cycles of $y(t)$ for each simulation run (more precisely, $T = 30\,000$ s for $\Omega = 1\ \mathrm{s}^{-1}$). The intensity D of the white-noise excitation was increased stepwise for each subsequent run with given α/Ω, β/Ω until a rather large value of $x(T)$ was attained, with extremely small values of $x(T)$ being observed for all smaller D's. The stepwise increase in D corresponded to resolution 0·01 in a non-dimensional parameter $\sigma_* = \sqrt{k^2 D/4\alpha\Omega^2}$ of the threshold excitation intensity. (This parameter is seen to be the r.m.s. value of a non-dimensional displacement Y/R of the corresponding linear system). There were no ambiguities in discriminating between values of $x(T)$ within and outside of the stability domain (with differences being about several orders of magnitudes), so that this procedure resulted in a sample stochastic stability boundary indeed.

The resulting values of σ_* as obtained for several different damping ratios α/Ω and $\beta = \alpha$ are presented in Table 1 (it seems that damping should typically be the same for X and Y directions in axially symmetric systems). A very strong influence of damping on the stability threshold can be seen, which usually is typical for stochastic stability. In the present case the influence is twofold: firstly, increasing damping reduces the response in the Y direction, thereby reducing the level of parametric excitation, and secondly, it provides a stabilizing effect for the X direction.

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